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# An analytical study of the metric dimension in context of wheel related graphs

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### Abstract
An ordered set $W = (w_1, \dots, w_k) \subseteq V(G)$ vertices of $G$ is called a resolving set or locating set for $G$ if every vertex is uniquely determined by its vector of distance to the vertices in $W$. A resolving set of minimum cardinality is called a basis for $G$ and this cardinality is the metric dimension or location number of $G$, denoted by $\beta(G)$. In this paper, we study the metric dimension of certain wheel related graphs, namely m-level wheels, an infinite class of convex polytopes and antiweb-gear graphs denoted by $W_{n,m}, \mathbb{Q}$ and $AWJ_{2n}$, respectively. We prove that these infinite classes of convex polytopes generated by wheel, denoted by $\mathbb{Q}_n$ also gives a negative answer to an open problem proposed by Imran *et al*. (2012).

**Keywords:** metric dimension, context, wheel related graphs

### Introduction
Metric dimension is a parameter that has appeared in various applications of graph theory, as diverse as, pharmaceutical chemistry robot navigation combinatorial optimization networking, facility location problems and sonar and coast guard Loran to name a few. The metric dimension of graph has been found of key importance of the evolution of cooperation. A basic problem in chemistry is to provide mathematical representation for a set chemical compound can be represented by a labeled graph whose vertex and edge labels specify the atom and bond types, respectively. This, a graph-theoretic interpretation of this problem is to provide representation for the vertex of a graph in such a way that distinct vertices have distinct representation. The distance $d(u, v)$ between two vertices $u, v \in V(G)$ in a connected graph $G$ is the length of a shortest path between them, while diameter of $G$, denoted by $diam(G)$ is the maximum distance between any pair of vertices $u, c \in V(G)$. Let $W = \{w_1, \dots., w_k\}$ be an ordered set of vertices of $G$ and let $v$ be a vertex of. the representation $r(v|W) of\ v$ with respect to $W$ is the $k - tupil$ $(d(v, w_1), d(v, w_2), \dots, d(v, w_k))$. If distinct vertices of $G$ have distinct representations with respect to $W$, then $W$ is called a resolving set or locating set for $G$ [3]. A resolving set of minimum cardinality is called a basis for $G$ and his cardinality is the metric dimension of $G$, denoted by $\beta(G)$. The concept of resolving sets and metric dimension have previously appeared in the literature.
For a given ordered set of vertex $W = \{w_1, w_2, \dots, w_k\}$ of a graph $G$, then $i^{th}$ component if $r(v|W)$ is 0 only if $w_i = v$. Thus, to show that W is a resolving set it suffices to verify that $r(x|W)$ for each pair of distinct vertices $x, y \in V(G) \backslash W$.
A useful property in finding the $\beta(G)$ is the following lemma.
Let $W$ be a resolving set for a connected graph $G$ and $u, v \in V(G)$. If $d(u, w) = d(v, w)$ for all vertex $w \in V(G)\{u, v\}, then\ \{u, v\} \cap W \neq \emptyset$.
Let $\mathcal{F}$ denotes family of connected graphs $G_n\colon \mathcal{F} = (G_n)_{n \geq 1}$ depending on $n$ as follows: the order $|V(G)| = \varphi(n)$ and $\lim_{n \to \infty} \varphi(n) = \infty$. if there exist a constant $C > 0$ such that $\beta(G_n) \leq C$ for every $n \geq 1$, then we shall say that $\mathcal{F}$ has bounded metric dimension; otherwise $\mathcal{F}$ has unbounded metric dimension.
If all graphs in $\mathcal{F}$ have the same metric dimension (which does not depend on $n$), $\mathcal{F}$ is called a family with constant metric dimension. The families of graph with constant metric dimension were discussed previously in then metric dimension of several classes of vertex

polytopes has been discussed in In this paper, we study the metric dimension of certain wheel related graphs, namely m-level wheel, an infinity class of convex polytopes defined in and antiweb-gear graphs denoted by $W_{n,m}, \mathbb{Q}$ $and$ $AWJ_{2n}$, respectively. The study of an infinity class of vertex polytopes gives a negative answer to an open problem proposed in We prove that these infinity classes of wheel related graphs have unbounded metric dimension.

**Metric dimension of m-level wheels**

Denoting by $G + H$ as join of two graphs, a wheel graphs denoted by $W_{n,1} \cong C_{n,1} + K_1$, where $C_{n,1}: v_1, v_2, \dots, v_n, v_1$ $for$ $n \geq 3$ is a cycle of length n. For our convenience, we denote the outer cycle of the wheel by $C_{n,1}$. It is provided in [3] that $\beta(W_{n,1}) = \left\lceil \frac{2n+2}{5} \right\rceil$ $for$ $m \geq 7$, imply that wheels have unbounded metric dimension.

Suppose $C_{n,1}$ is an outer cycle of length $n$ $of$ $W_{n,1}$. If B is a basis of $W_{n,1}$ then it contains $r \geq 2$ vertices on $C_{n,1}$ $for$ $n \geq 3$ and we can order the certices of $B = \{v_{i_1}, v_{i_2}, \dots, v_{i_r}\}$ so that $i_1 < i_2 \dots, i_r$. We shall say that the paris of vertices $\{v_{i_a}, v_{i_{a+1}}\}$ for $1 \leq a \leq r-1$ and $\{v_{i_r}, v_{i_1}\}$ are pairs of neighboring vertices. Given such an oedering, as in [3] we will define the gap $g_a$ for $a \leq a \leq r-1$ as the set vertices $\{v_j | i_a < j < i_{a+1}$ and $g_r = \{v_j | 1 \leq j \leq i_1 or\ i_r < j \leq n\}$.

Thus we have r gaps, some of which may be empty. We will say that gap $g_a$ and $g_b$ are neighboring gaps when $|a-b| = 1$ $or$ $r-1$. It was shown in that if B is a basis for $W_{n,1}$ then B consists only the vertices of $C_{n,1}$ that satisfy the following properties:

a) Every gap B contains at most three vertices.
b) At most one gap of B contains three vertices.
c) If a gap of B contains at least two vertices, then both of its neighboring gaps contain at most one vertex.

**Definition:** A double-wheel graph $W_{n,2}$ can be obtained as join of $2C_n + K_1$, and inductively we can construct an m-level wheel graph denoted by $W_{n,m} \cong mC_n + K_1$.

Let $C_{n,1}, \dots, C_{n,m}$ represent the cycle of $W_{n,m}$. At levels 1,…,m, respectively, as shown in fig.1. We want to compute the metric dimension of $W_{n,2}, \dots, W_{n,m}$. For this study the metric dimension of $W_{n,1}$.

Suppose that $W_{n,2} = 2C_{n,1} + K_1$ $fo$ $n \geq 3$, then the central vertex $v$ does not belong to any basis. Since $diam(W_{n,2}) = 2$, so if $v$ belongs to any metric basis, say B, then there must exist two distinct vertices $v_i and\ v_j$, for $1 \leq i \neq j \leq n$ such that

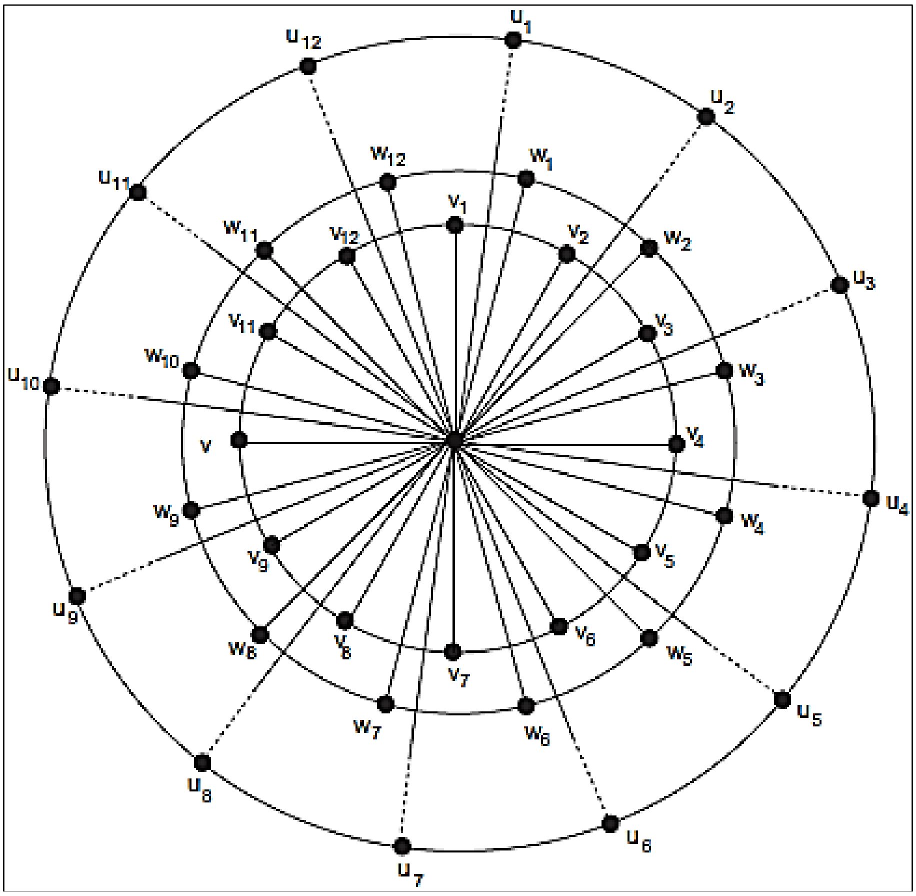


**Fig 1:** An m-level wheel $W_{12,m}$

$r(v_i | B = r(v_j | B)$. Consequenly, the basis vertices belong to the rim vertices of $W_{n,2}$ only if B is a basis of $W_{n,2}$, then contains only vertices from the cycle induced by $C_{n,1}$ $and$ $C_{n,2}$. We have the following gap conditions for the selection of basis vertices:

1. Every gap of B for the vertices of $C_{n,1}$ satisfy conditions $(a)-(c)$ presented for $W_{n,1}$.

2. Every gap of B may have at most three vertices of $C_{n,1} or\ C_{n,2}$. Otherwise, there may be a gap having three vertices, say, $W_i, w_{i+1}, w_{i+2} (1 \le i \le n) of\ C_{n,2}$ and addition performed modulo n such that $r(w_{i+1}|B = r(v_{i+1}|B), where\ v_i, v_{i+1}, v_{i+2}$ are the vertices of the gap of $C_{n,1}$. In other words, we can say that at most one gap of B have three vertices.
3. If a gap of B has two vertices then its neighboring gap contains at most one vertex. Otherwise, there exist five consecutive vertices, say, $w_i, w_{i+1}, w_{i+2}, w_{i+3}, w_{i+4}$ such that $w_{i+2} \in B\ (1 \le i \le n)$. However, then we have $r(w_{i+1}|B) = r(w_{i+3}|B)$.

Now suppose that B is any set of vertices of $C_{n,1} and\ C_{n,2}$ that satisfies conditions (i)-(iii) and let $y \in V(W_{n,2})\backslash B$. there are following possibilities to be discussed:
1. If $y$ belongs to gap of B of vertices of $C_{n,1}$ the it must satisfy the following conditions:
a) $y$ belongs to a gap of size one of B. suppose $v_i and\ v_j$ be the neighboring vertices of B thaht determine this gap. Then $y$ is adjacent to $v_i and\ v_j$ and has distance two from all other vertices of B. since $n \ge 7$, no other vertices of $W_{n1}$ has this property and so $r(y|B) \ne (x|B) for\ x \ne\ y$.
b) $y$ Belongs to a gap of size two of B. then we may assume that $v_j, v_{j+1} = y, y_{j+2}, y_{j+3}$ are vertices of $C_{n,1}$, where $v_{j+1}, v_{j+3} \in B$ and $v_{j+2} \notin B$. Then $y$ is adjacent to $v_j$ and has distance 2 from all other vertices of B. By property (C), only $y$ has this property and so $r(y|B) \ne r(x)|B) for\ x \ne\ y$
c) $y$ Belongs to a gap of size three of B. Then there exists vertices $v_j, v_{j+1}, v_{j+2}, v_{j+2}, v_{j+3}, v_{j+4}\ of\ C_{n,1}$, where only $v_{j+1}, v_{j+4} \in B$. Assume first that $y = v_{j+1}$. Then $y$ adjacent to $v_j$ and has distance 2 from all other vertices of B. by property ©, $y$ is the only vertex of $W_{n,1}$ with this property and so $r(y|B) \ne r(x|B) for\ x \ne y$.
Next, we assume that $y = v_{j+2}$. Then$r(y|B) = 2,2,\dots,2)$. By properties (a) and (b), no other vertex of $W_{n,1}$ has this representation.
d) $y = v$ be a central vertex. Then $r(y|B) = 1,1,\dots,1)\ and\ y$ is the only vertex of $W\_(n,1)$ with representation.
2. Similarly, one can show that if either y belongs to a gap of size one, two or three of B of vertices of cycle induced by $C_{n,2} or\ if\ y$ is a central vertex of $W_{n,2}$ then, we have $r(y|B) \ne r(x|B)$ for $x \ne y; y \in V(W_{n,2})$.

**Therefore, any set B having properties (i)-(iii) is a resolving set for $W_{n,2}$**

In the next theorem, we give a precise formula for computing the metric dimension of double wheel $W_{n,2}$ for $n \ge 7$. this result provides a base for extending the result to the metric dimension of m-level wheels.

Theorem 2.1 $If\ n \ge 7$, then we have $\beta(W_{n,2}) = \beta(W_{n,1}) + \left\lfloor\frac{2n+4}{5}\right\rfloor$.

Proof. Let $W_{n,1} \cong C_{n,1} + K$ and $W_{n,2} \cong 2C_{n,1} + K_1$, where $v$ is the central vertex of $W_{n,2}$ and $C_{n,1}: v_1, \dots, v_n, v_1$ and $C_{n,2}: w_1, \dots, w_n, w_1$ be the outer cycles of $W_{n,2}$ at levels 1 and 2, respectively. First we prove that $\beta(W_{n,2}) \le \beta(W_{n,1}) + \left\lfloor\frac{2n+4}{5}\right\rfloor$ by constructing a resolving set in $W_{n,2}$ with $\beta(W_{n,1}) + \left\lfloor\frac{2n+4}{5}\right\rfloor$ vertices. We assume the following cases according to the residue class modulo 5 to which n belongs.

**Case 1:** When $n \equiv 0\ (mod\ 5)$, the we may write $n = 5k$, where $k \ge 2$, and $\beta(W_{n,1}) + \left\lfloor\frac{2n+4}{5}\right\rfloor = 4k$. since $B \equiv \{v_{5i+1}, v_{5i+4}, w_{5j+1}, w_{5j+4}: 0 \le i, j \le k-1$, it is resolving set having 4k vertices as it satisfies conditions (i)-(iii).

**Case 2:** When $n \equiv 1(mod\ 5)$, then we may write $n = 5k+1\ where\ k \ge 2, and\ \beta(W_{n,1}) + \left\lfloor\frac{2n+4}{5}\right\rfloor = 4k+1$. Since $B = \{v_{5i+1}, v_{5i+4}: 0 \le i \le k-2\} \cup \{v_{5k-4}, v_{5k}\} \cup \{w_{5j+1}, w_{5j+4}: 0 \le j \le k-1\} \cup \{w_{5k+1}\}$, it is a resolving set having $4k+1$ vertices as it satisfies conditions (i)-(iii).

**Case 3:** When $n \equiv 2(mod\ 5)$, then we may write $n = 5k+2\ where\ k \ge 1, and\ \beta(W_{n,1}) + \left\lfloor\frac{2n+4}{5}\right\rfloor = 4k+2$. Since $B = \{v_{5i+1}, v_{5i+4}, w_{5j+1}, w_{5j+4}: 0 \le i, j \le k-1\} \cup \{v_{5k+1}, v_{5k+1}\}$,t is a resolving set having $4k+2$ vertices as it satisfies conditions (i)-(iii).

**Case 4:** When $n \equiv 4\ (mod\ 5)$, then we may write $n = 5k+4\ where\ k \ge 1, and\ \beta(W_{n,1}) + \left\lfloor\frac{2n+4}{5}\right\rfloor = 4k+4$. Since $B = \{v_{5i+1}, v_{5i+4}, w_{5j+1}, w_{5j+4}: 0 \le i, j \le k\}$. It is a resolving set having $4k+4$ vertices as it satisfies conditions (i)-(iii).

**Case 5:** When $n \equiv 3\ (mod\ 5)$, then we may write $n = 5k+3\ where\ k \ge 1, and\ \beta(W_{n,1}) + \left\lfloor\frac{2n+4}{5}\right\rfloor = 4k+3$. Since $B = \{v_{5i+1}, v_{5i+4},: 0 \le i \le k-2\} \cup \{v_{5k-4}, v_{5k,} v_{5k+2}\} \cup \{w_{5j+4}, w_{5j+6}: 0 \le j \le k-1\} \cup \{w_1, w_{5k+3}\}$,t is a resolving set having $4k+3$ vertices as it satisfies conditions (i)-(iii).

Hence, it follows from above discussion that $\beta(W_{n,2}) \le \beta(W_{n,1}) + \left\lfloor\frac{2n+4}{5}\right\rfloor$.

Next, we show that $\beta(W_{n,2}) \ge \beta(W_{n,1}) + \left\lfloor\frac{2n+4}{5}\right\rfloor$. Let B a basis for $W_{n,2}$. We consider the following cases:

Case (a). subcase($a_1$): $|B_1| = 2l$ for some integer $l \ge 1$, where $B_1$ is the basis for $W_{n,1}$ as obtained in [3], i.e. $|B_1| \ge \beta(W_{n,1}) = \left\lfloor\frac{2n+2}{5}\right\rfloor$.

Subcase ($a_2$): $|B_2|=2t$ for some integer $t \geq 1$, where $B_2$ represents the resolving vertices lying on $C_{n,2}$ in presence of vertices $B_1$. The conditions (i)-(iii) imply that at most t gaps of $B_2$ contain two vertices. So the number of vertices that belong to different gaps of $B_2$ are at most $3t$. Therefore we get, $n-2t \leq 3t$ which implies that $|B_2| = 2t \geq \left\lceil\frac{2n}{5}\right\rceil \geq \left\lfloor\frac{2n+4}{5}\right\rfloor$.
Subcase($a_3$): $|B_2| = 2t+1$ for some integer $t \geq 1$, where $B_2$ represents the resolving vertices lying on $C_{n,2}$ in presence of vertices $B_1$. Condition (i)-(iii) implies that at most t gaps of $B_2$ contains two vertices. So the number of vertices that belong to different gaps of $B_2$ are at most $3t+1$. Therefore we get, $n-2t-a \leq 3t+1$ which implies that $|B_2| = 2t+1 \geq \left\lceil\frac{2n+1}{5}\right\rceil \geq \left\lfloor\frac{2n+1}{5}\right\rfloor$. Hence by combining subcase($a_1$) with subcase ($a_2$) or subcase ($a_3$), we obtain that $|B| = |B_1| + |B_2| \geq 2l + 2t \geq \beta(W_{n,1}) + \left\lceil\frac{2n+4}{5}\right\rceil$.
Case (b). subcase($b_1$): $|B_1| = 2l+1$ for some integer $l \geq 1$, where $B_1$ is the basis for $W_{n,1}$ as obtained in [3], i.e. $|B_1| \geq \beta(W_{n,1})$.
Subcase($b_2$): $|B_2| = 2t+1$ for some integer $t \geq 1$, where $B_2$ represents the resolving vertices lying on $C_{n,2}$ in presence of vertices $B_1$. Condition (i)-(iii) implies that at most t gaps of $B_2$ contain two vertices. So the number of vertices that belong to different gaps of $B_2$ are at most $3t+1$. therefore we get, $n-2t-1 \leq 3t+1$ which implies that $|B_2| = 2t+1 \geq \left\lceil\frac{2n+1}{5}\right\rceil \geq \left\lfloor\frac{2n+4}{5}\right\rfloor$.
Subcase ($b_3$): This case is similar to the cubase $(a_2)$. Hence by combining subcase ($b_1$) with subcase ($b_2$) or subcase ($b_3$), we obtain that $|B| = |B_1| + |B_2| \geq 2l+1+2t+1 \geq \beta(W_{n,1}) + \left\lfloor\frac{2n+4}{5}\right\rfloor$, which completes the proof.
We now extend out results to m-level wheel denoted by $W_{n,m}$ In the next theorem, we apply mathematical induction on the levels of wheel to prove result.
Theorem 2.2 we have $\beta(W_{n,m})\beta(W_{n,1}) + (m-1)\left\lfloor\frac{2n+4}{5}\right\rfloor$ for every integer $n \geq 7$ and $m \geq 3$.
Proof. We will prove this result by induction on levels of wheel denoted by 'm'. When m=1, then $\beta(W_{n,1}) = \left\lfloor\frac{2n+2}{5}\right\rfloor$ is obtained in [3]. When m=2, then $\beta(W_{n,2}) = \beta(W_{n,1}) + \left\lfloor\frac{2n+4}{5}\right\rfloor$ by theorem 2.1. Now we assume that the assertion is true for $m = k$ i.e.,

$$\beta(W_{n,k}) = \beta(W_{n,1}) + (k-1)\left\lfloor\frac{2n+4}{5}\right\rfloor. \tag{1}$$

We will shoe that it is true for $m = k+1$. Suppose $\beta(W_{n,k+1}) = \beta(W_{n,k}) + \left\lfloor\frac{2n+4}{5}\right\rfloor$, then by using Eq. (1), we have $\beta(W_{n,k+1}) = \left\{\beta(W_{n,1}) + (k-1)\left\lfloor\frac{2n+4}{5}\right\rfloor\right\} + \left\lfloor\frac{2n+4}{5}\right\rfloor = \beta(W_{n,1}) + (k)\left\lfloor\frac{2n+4}{5}\right\rfloor$. Hence the result is true for all positive integers $m \geq 3$.

## 1. Metric dimension of an infinity class of convex polytopes

Let $I = (1, \dots, n\}$ be an index set and $Q_n$ be the graph of an antiprism. The antiprism $Q_n, n \geq 3$ [16] is the plane regular graph. Let us denote the vertex set of $Q_n$ by $V(Q_n) = \{y_1, y_2, \dots, y_n, z_1, z_2, \dots, z_n\}$ and the edge set by $E(Q_n) = \{y_i y_{i+1} : i \in I\} \cup \{z_i z_{i+1} : i \in I\} \cup \{y_i z_i + 1 : i \in I\}$. We make the convention that $y_{n+1} = y_1$ and $z_{n+1} = z_1$ to simplify later notations. The face set $F(Q_n)$ contains $2n$ 3-sided face and two n-sided face(internal and external). We insert exactly one vertex $x(t)$ into the internal (external) n-sided face of $Q_n$ and consider the graph $\mathbb{Q}_n$ with the vertex set $V(\mathbb{Q}_n) = V(Q_n) \cup \{x, t\}$ and the edge set $E(\mathbb{Q}_n) = E(Q_n) \cup \{x y_{i+1} : i \in I\} \cup \{z_i t : i \in I\}$ the $\mathbb{Q}_n$ is the plane graph consisting of 3-sided faces and constitutes an infinite class of convex polytopes.
The metric dimension of several classes of graphs was studied in [8, 9, 12, 13, 15-17] and was proved that the classes of convex polytopes have constant metric dimensions. The following open problem was proposed in [8].
Open Problem is it the case that graph of every convex polytope has constant metric dimension? In this section, we study the metric dimension of this class of convex polytopes denoted by $\mathbb{Q}_n$ and we prove that this class of graph has unbounded metric dimension, thus giving a negative answer to the open problem proposed in [9]. Let $y_1, y_2, \dots, y_n and\ z_1, z_2, \dots, z_n$ represented the vertices of inner cycle $C_{n,1}$ an outer cycle $C_{n,2} of\ \mathbb{Q}_n$ respectvely as shown in fig.2. Suppose that $\mathbb{Q}_n$ for $n \geq 3$ be an infinite class of convex polytopes, then the central vertices $x\ and\ t$ do not belongs to any basis. Since $diam(\mathbb{Q}_n) = 3$, so if one of $c\ and\ t$ belongs to any metric basis, say B, then there must exist two distince vertices $v_i and\ v_j$, for $11 \leq i \neq j \neq n$ such that $r(v_i|B) = r(v_j|B)$. Consequently, the basis vertices belong to the rim vertices of $\mathbb{Q}_n$ only.
If B is a basis of $\mathbb{Q}_n$, then B contains only vertices of inner cycle of $\mathbb{Q}_n$. We have the following gap conditions for the selection of the basis vertices:

i) Every gap of B may have at most two vertices of $C_{n,1}$. Otherwise there exist a gap of B having three vertices $y_p, y_{p+1}, y_{p+2}, y_{p+3}\ and\ y_{p+4}\ with\ y_p, y_{p+4} \in B$ such that $r(y_{p+2}|B) = r(t|B) = (2,2, \dots, 2)$.
ii) If a gap of B contains two vertices of $C_{n,1}$, then its neighboring gaps may contain at most one vertex. Otherwise, there exist five consecutive vertices $y_p, y_{p+1}, y_{p+2}, y_{p+3}\ and\ y_{p+4}\ with\ y_{p+2} \in B$ such that $r(y_{p+1}|B) = r(y_{p+3}|B)$.

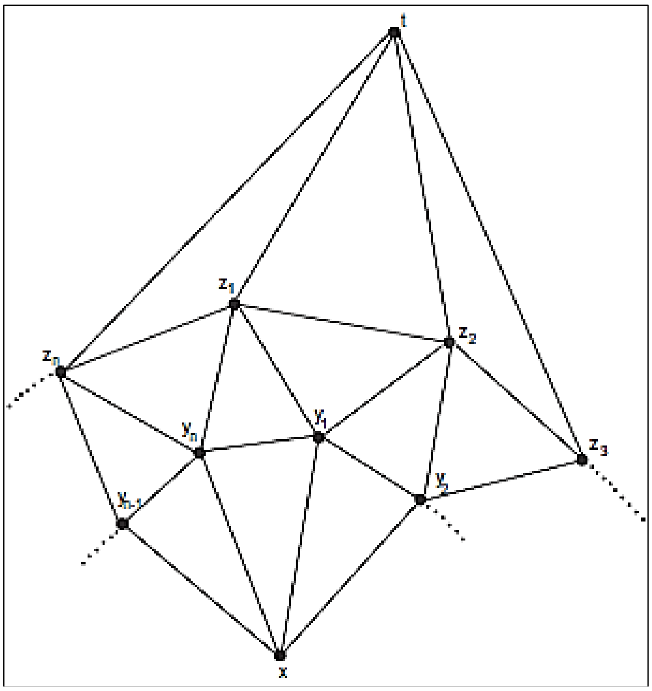


**Fig 2:** An infinite class of convex polytopes $\mathbb{Q}_n$

Now we assume that B is any set of vertices of $C_{n,1}$ that satisfies condition (i) and (ii) and let $y \in V(\mathbb{Q}_n)$. There are following possibilities to be discussed:

- If $y$ belongs to a gap of size two of B with vertices $y_p, y_{p+1} = y, y_{p+2}, y_{p+3}$ such that $y_p, y_{p+3} \in B$, then $r(y|B) = (1,2, \dots, 2)$.
- If $y$ belongs to a gap of size two of B with vertices $y_p, y_{p+1} = y, y_{p+2}$ such that $y_p, y_{p+2} \in B$, then $r(y|B) = (1,1,2, \dots, 2)$.
- If $y = t$ then $r(y|B) = (2,2, \dots, 2)$.
- If $y = x$ then $r(y|B) = (1,1, \dots, 1)$.
- If $y = z_p \in V(C_{n,2})$ and $y$ is adjacent to $y_p$ and $y_{p+n-1}$ with $y_p, y_{p+3}, y_{p+n-1} \in B$, then $r(y|B) = (1,3, \dots, 3,1)$.
- If $y = z_p \in V(C_{n,2})$ and $y$ is adjacent to $y_p$ and $y_{p+n-1}$ with $y_p, y_{p+3}, y_{p+n-2} \in B$, then $r(y|B) = (1,3, \dots, 3,2)$.
- If $y = z_p \in V(C_{n,2})$ and $y$ is adjacent to $y_p$ and $y_{p+n-1}$ with $y_{p+1}, y_{p+n-2} \in B$, then $r(y|B) = (2,3, \dots, 3,2)$.
- If $y = z_p \in V(C_{n,2})$ and $y$ is adjacent to $y_p$ and $y_{p+n-1}$ with $y_{p+1}, y_{p+n-3}, y_{p+n-1} \in B$, then $r(y|B) = (2,3, \dots, 3,1)$.

Therefore, any set B having properties (i) and (ii) is a resolving ser of $\mathbb{Q}_n$. We now present an exact formula for computing the metric dimension of $\mathbb{Q}_n$ for every integer $n \geq 6$.

Theorem 3.1. $If\ n \geq 6$, then we have $\beta(\mathbb{Q}_n) = \left\lfloor \frac{2n+4}{5} \right\rfloor$.

Proof. We prove this result by double inequality. First we prove that $\beta(\mathbb{Q}_n) \leq \left\lfloor \frac{(2n+4)}{5} \right\rfloor$ by constructing a resolving set in $\mathbb{Q}_n$ with $\left\lfloor \frac{2n+4}{5} \right\rfloor$ vertices. We consider the following cases according to the residue class modulo 5 to which $n$ belongs.

**Case 1:** When $n \equiv 0\ (mod\ 5)$, then we may write $n = 5k$, where $k \geq 2$, and $\left\lfloor \frac{2n+4}{5} \right\rfloor = 2k$. Since $B = \{y_{5i+1}, y_{5i+4} : 0 \leq i \leq k-1\}$, it is resolving set having $2k$ vertices as it satisfies conditions (i) and (ii).

**Case 2:** When $n \equiv 1\ (mod\ 5)$, then we may write $n = 5k+1$, where $k \geq 1$, and $\left\lfloor \frac{2n+4}{5} \right\rfloor = 2k+1$. Since $B = \{y_{5i+1}, y_{5i+4} : 0 \leq i \leq k-1\} \cup \{y_{5k+1}\}$, it is resolving set having $2k+1$ vertices as it satisfies conditions (i) and (ii).

**Case 3:** When $n \equiv 2\ (mod\ 5)$, then we may write $n = 5k+2$, where $k \geq 1$, and $\left\lfloor \frac{2n+4}{5} \right\rfloor = 2k+1$. Since $B = \{y_{5i+1}, y_{5i+4} : 0 \leq i \leq k-1\} \cup (w_{5k+1}$, it is resolving set having $2k+1$ vertices as it satisfies conditions (i) and (ii).

**Case 4:** When $n \equiv 3\ (mod\ 5)$, then we may write $n = 5k+3$, where $k \geq 1$, and $\left\lfloor \frac{2n+4}{5} \right\rfloor = 2k+2$. Since $B = \{y_{5i+4}, y_{5i+6} : 0 \leq i \leq k-1\} \cup (y_1, y_{5k+3}$, it is resolving set having $2k+2$ vertices as it satisfies conditions (i) and (ii).

**Case 5:** When $n \equiv 4\ (mod\ 5)$, then we may write $n = 5k+4$, where $k \geq 1$, and $\left\lfloor \frac{2n+4}{5} \right\rfloor = 2k+2$. Since $B = \{y_{5i+1}, y_{5i+4} : 0 \leq i \leq k\}$, it is resolving set having $2k+2$ vertices as it satisfies conditions (i) and (ii). Hence, from above it follows that $\beta(\mathbb{Q}_n) \leq \left\lfloor \frac{2n+4}{5} \right\rfloor$.

Next we show that $\beta(\mathbb{Q}_n) \geq \left\lfloor \frac{2n+4}{5} \right\rfloor$. Let B be a basis of $\mathbb{Q}_n$. We consider the following cases:

**Case (a):** $|B| = 2t$ for some integer $t \geq 1$. the conditions (i) and (ii) imply that at most $t$ gaps of B contains two vertices. So the number of vertices that belong to different gaps of B are at most $3t$. Therefore $n = 2t \leq 3t$, which implies that $|B| = 2t \geq \left\lceil \frac{2n}{5} \right\rceil \geq \left\lfloor \frac{2n+4}{5} \right\rfloor$.

**Case (b):** $|B| = 2t + 1$ for some integer $t \geq 1$. the conditions (i) and (ii) imply that at most $t$ gaps of B contains two vertices. So the number of vertices that belong to different gaps of B are at most $3t + 1$. Therefore $n - 2t - 1 \leq 3t + 1$, which implies that $|B| = 2t + 1 \geq \left\lceil \frac{2n+1}{5} \right\rceil \geq \left\lfloor \frac{2n+4}{5} \right\rfloor$. Which complete the proof.

**2. Metric dimension of antiweb-gear graphs**

The gear graph denoted by $j_{2n}$ is defined as follows: Consider an even cycle $C_{2n}: v_1, v_2, \dots, v_{2n}, v_1$ where $n \geq 2$ and a new vertex $v$ is adjacent to $n$ vertices of $C_{2n}: v_2, v_4, \dots, v_{2n}$.

The gear graph $J_{2n}$ can be obtained from the wheel $W_{2n}$ by alternately deleting $n$ spokes. Tomescu and Javaid [33] proved that $\backslash bata\ (J_{2n}) = \left\lfloor \frac{2n}{3} \right\rfloor\ for\ n \geq 4$.

An antiweb-wheel denoted by $AWW_n$ can be defined as $AWW_n \cong C_n^2 + K_1$. We have $V(AWW_n) = V(W_n)$ and $E(AWW_n) = E(W_n) \cup \{v_i v_{i+2}: 0 \leq i \leq n\}$, where the indices are taken modulo $n$. In [23], it was proved that

$$\beta(AWW_n) = \begin{cases} \left\lceil \frac{n+1}{3} \right\rceil & : if\ n\ is\ odd; \\ \frac{n}{3} & : otherwise. \end{cases}$$

The antiweb-gear graph can be obtained from gear graph $J_{2n}$ by replacing $C_{2n}$ by $C_{2n}^2$ and is denoted by $AWJ_{2n}$. We have $V(AWJ_{2n}) = V(J_{2n})$ and $E(AWJ_{2n}) = E(J_{2n}) \cup \{v_i v_{i+2}: 0 \leq i \leq n$, where the indices are taken modulo $n$. In this section, we study the metric dimension of antiweb-gear graphs and we prove that this class has unbounded metric dimension.

Suppose that $AWJ_{2n}$ for $n \geq 3$, then the central vertex $v$ does not belong to any basis. Since $diam(AWJ_{2n}) = 4$, if $v$ belongs to any metric basis, say B, then there must exist two distinct vertices $v_i and\ v_j$ for $1 \leq i \neq j \leq n$ such that $r(v_i|B) = r(v_j|B)$. Consequently, the basis vertices belong to the rim vertices of $AWJ_{2n}$ only (An antiweb-gear graph is shown in fig. (3).

A gap determined by neighboring vertices $v_i\ and\ v_j$ be called an $\alpha - \beta$with $\alpha \leq \beta$ when $\deg(v_i) = \alpha$ and $\deg(v_j) = \beta$ or When $\deg(v_i) = \beta and \deg(v_j) = \alpha$. Hence we have three kinds of gaps in $AWJ_{2n}$, i.e. 4-4, 5-4 and 5-5 gaps.

Lemma 4.1 Let B be a basis of $AWJ_{2n} n \geq 6$, then every 4-4, 5-4 and 5-5 gap of B contains at most 9,8 and 7 vertices respectively.

Proof. On contrary, suppose that there is a 4-4 gap of B having 11 vertices $v_1, \dots, v_{11}\ of\ C_{2n}$ such that $\deg(v_1) = \deg(v_{11}) = 5$. For this case, $(v_5|B) = r(v_7|B)$, contradiction. Similarly, if there us a 5-4 having 10 vertices if $C_{2n}$ say, $v_1, \dots, v_{10}$such that $\deg(v_1) = 5$ and $\deg v(v_{10}) = 4$. In this case, we get $r(v_5|B) = r(v\_7|B)$, a contradiction. If there is a 5-5 gap having 9 vertices say, $v_1, \dots, v_9$ such that $\deg(v_1) = \deg(v_9) = 4$, then $r(v_5|B) = r(v_7|B)$, contradiction.

From now on, the 4-4, 5-4 and 5-5 gaps having 9, 8 and 7 vertices, respectively will be referred as major gaps, while the rest of all will referred as minor gaps. The vertices having degree 5 and 4 are known as major (labeled by star) and minor vertices, respectively.

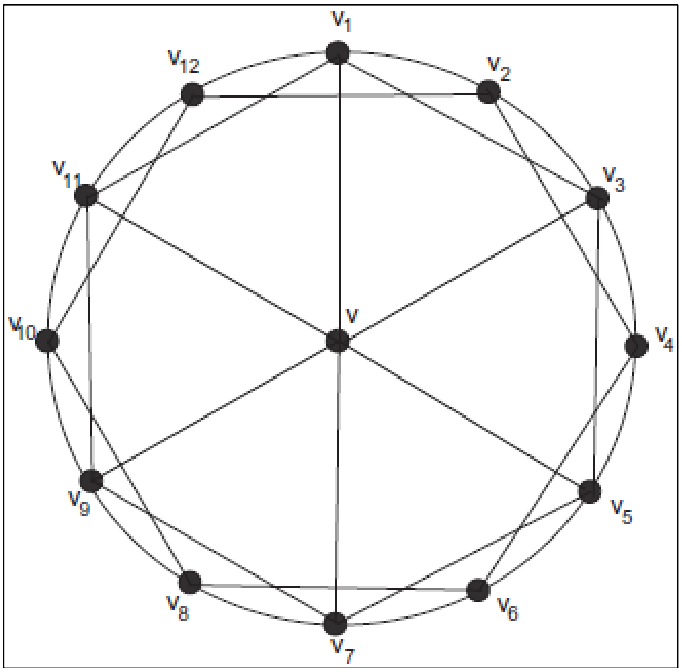


**Fig 3:** An antiweb-gear graph $AWJ_{12}$

Lemma 4.2 any basis B of $AWJ_{2n}$ $n \geq 6$) contains at most one major 4-4 or 5-4 gap.
Proof in contrary, suppose that B contains two distince major gaps of kind 4-4 or 5-4, then we have the following cases:

- 4-4 and 4-4 gaps: $\overset{*}{v}_1, v_2, \overset{*}{v}_3, v_4, \overset{*}{v}_5, v_6, \overset{*}{v}_7, v_8, \overset{*}{v}_9$ and $\overset{*}{u}_1, u_2, \overset{*}{u}_3, u_4, \overset{*}{u}_5, u_6, \overset{*}{u}_7, u_8, \overset{*}{u}_9$; in this case we have $r\left(\overset{*}{v}_5\middle|B\right) = r(\overset{*}{v}_7|B)$.
- 4-4 and 5-4 gaps: $\overset{*}{v}_1, v_2, \overset{*}{v}_3, v_4, \overset{*}{v}_5, v_6, \overset{*}{v}_7, v_8, \overset{*}{v}_9$ and $u_1, \overset{*}{u}_2, u_3, \overset{*}{u}_4, u_5, \overset{*}{u}_6, u_7, \overset{*}{u}_8$ ; in this case we have $r\left(\overset{*}{v}_5\middle|B\right) = r(\overset{*}{v}_4|B)$.
- 5-4 and 5-4 gaps: $u_1, \overset{*}{u}_2, u_3, \overset{*}{u}_4, u_5, \overset{*}{u}_6, u_7, \overset{*}{u}_8$ and $v_1, \overset{*}{v}_2, v_3, \overset{*}{v}_4, v_5, \overset{*}{v}_6, v_7, \overset{*}{v}_8$; in this case we have $r\left(\overset{*}{v}_4\middle|B\right) = r\left(\overset{*}{u}_4\middle|B\right)$.

In the next lemma, we will prove that any two neighboring gaps, one of which being major may contain together at most 12 vertices.
Lemma 4.3. For any basis B of $AWJ_{2n}$ $(n \geq 6)$, any two neighboring gaps, one of which being major of kind4-4, 5-4 or 5-5 contain together at most 12 vertices.
Proof. If the major gap is a 4-4 (with 9) vertices, then by Lemma 4.2 its neighboring gap can neither be a 4-4 gap having 5 vertices nor be a 5-4 gap having 4 vertices. If it is true, consider a path: $\overset{*}{v}_1, v_2, \overset{*}{v}_3, v_4, \overset{*}{v}_5, v_6, \overset{*}{v}_7, v_8, \overset{*}{v}_9, v_{10}, \overset{*}{v}_{11}, v_{12}, \overset{*}{v}_{13}, v_{14}, \overset{*}{v}_{15}$ on $C_{2n}$ where $v_{10} \in B$ such that $r(v_9|B) = r(v_{11}|B)$, contradiction. If the major gap is a 5-4 gap (with 8) vertices then by lemma 4.2, its neighboring gap can't be a 4-4 gap having 5 vertices. If it is true, consider a path
$v_1, \overset{*}{v}_2, v_3, \overset{*}{v}_4, v_5, \overset{*}{v}_6, v_7, \overset{*}{v}_8, v_9, \overset{*}{v}_{10}, v_{11}, \overset{*}{v}_{12}, v_{13}, \overset{*}{v}_{14}$ on $C_{2n}$where $v_9 \in B$ such that $r(v_8|B) = r(v_{10}|B)$, contradiction. If the major gap is a 5-5 gap having 7 vertices then its neighboring gap can't be a minor 5-5 gap having 5 vertices. If it is true; consider a path: $v_1, \overset{*}{v}_2, v_3, \overset{*}{v}_4, v_5, \overset{*}{v}_6, v_7, \overset{*}{v}_8, v_9, \overset{*}{v}_{10}, v_{11}, \overset{*}{v}_{12}, v_{13}$ on $C_{2n}$ where $v_8 \in B$ such that $r\left(\overset{*}{v}_6\middle|B\right) = r(\overset{*}{v}_{10}|B)$, contradiction.
In the next lemma, we will prove that any two minor neighboring gaps may contain together at most e=ten vertices.
Lemma 4.4. If B is any basis of $AQJ_{2n}$ $(n \geq 6)$, then any two minor neighboring gaps contain together at most 10 vertices.
Proof. By lemma 4.1, any minor 4-4, 5-4 and 5-5 gap contains 7,6 and 5 vertices, respectively, it suffices to prove the following cases:

- Any minor 4-4 gap having 5 or 7 vertices has a neighboring 4-4 or 5-4 gaps with at most 3 and 2 vertices, respectively. Otherwise, there is a neighboring 4-4 or 5-4 gap having 5 and 4 vertices, respectively. In this case, consider a path: $\overset{*}{v}_1, v_2, \overset{*}{v}_3, v_4, \overset{*}{v}_5, v_6, \overset{*}{v}_7, v_8, \overset{*}{v}_9, v_{10}, \overset{*}{v}_{11}, v_{12}, \overset{*}{v}_{13}$ on $C_{2n}$, where $v_8 \in B$ such that $r\left(\overset{*}{v}_7\middle|B\right) = r(\overset{*}{v}_9|B)$, a contradiction.
- Any minor 5-4 gap having 6 or 4 vertices has a neighboring 4-4 or 5-4 gaps with at most 3 and 4 vertices, respectively. Otherwise, there is a neighboring 4-4 or 5-4 gap having 5 and 6 vertices, respectively. In this case, consider a path: $v_1, \overset{*}{v}_2, v_3, \overset{*}{v}_4, v_5, \overset{*}{v}_6, v_7, \overset{*}{v}_8, v_9, \overset{*}{v}_{10}, v_{11}, \overset{*}{v}_{12}$ on $C_{2n}$, where $v_7 \in B$ such that $r\left(\overset{*}{v}_6\middle|B\right) = r(\overset{*}{v}_8|B)$. Or $\overset{*}{v}_1, v_2, \overset{*}{v}_3, v_4, \overset{*}{v}_5, v_6, \overset{*}{v}_7, v_8, \overset{*}{v}_9, v_{10}, \overset{*}{v}_{11}, v_{12}, \overset{*}{v}_{13}$ on $C_{2n}$, where $v_7 \in B$ such that $r(v_6|B) = r(v_8|B)$, a contradiction.
- Any 5-5 gap having 5 vertices has a neighboring 5-5 or 5-4 gap with most 3 and 4 vertices, respectively. Otherwise, the neighboring gaps may contain 5 and 6 vertices, respectively. In this case, consider a path: $v_1, \overset{*}{v}_2, v_3, \overset{*}{v}_4, v_5, \overset{*}{v}_6, v_7, \overset{*}{v}_8, v_9, \overset{*}{v}_{10}, v_{11}, \overset{*}{v}_{12}, v_{13}$ on $C_{2n}$, where $\overset{*}{v}_7 \in B$ such that $r(v_5|B) = r(v_7|B)$, contradiction.

In the next theorem, we compute the exact value of metric dimension for antiweb-gear graph.
Theorem 4.1. For every $n \geq 15$, we have $\beta(AWJ_{2n}) = \left\lceil\frac{n+1}{3}\right\rceil$.
Proof. Consider the antiweb-gear graphs $AWJ_{2n}$, then we have $\beta(AWJ_{2n}) = 3$, for all $2 \leq n \leq 8$ and $W_1 = \{v_1, v_2, v_3\}$ and $W_2 = \{v_1, v_4, v_9\}$ being metric basis for all $2 \leq n \leq 6$ and $n = 7,8$ respectively. $\beta(AWJ_{2n}) = 4$, for all $9 \leq n \leq 12$ and $W_3 = \{v_1, v_4, v_8, v_{15}\}, W_4 = \{v_1, v_2, v_{10}, v_{12}\}$, $W_5 = \{v_1, v_2, v_{10}, v_{12}\}$ and $W_6 = \{v_1, v_4, v_{12}, v_{16}\}$ being metric basis for $n = 9, 10, 11, 12$, respectively. $\beta(AWJ_{2n}) = 5$ and $W_7 = \{v_1, v_5, v_{12}, v_{15}, v_{20}\}$ and $W_8 = \{v_1, v_4, v_{12}, v_{16}, v_{20}\}$ being metric basis for $n = 13, 14$, respectively. However for $n \geq 15$, the dimension of $AWJ_{2n}$ increases with number of vertices $n$. We also know that central vertex can't belong to any basis of $AWJ_{2n}$. For $n \geq 15$, we prove the result by double inequality. First we show that $\beta(AWJ_{2n}) \leq \left\lceil\frac{n+1}{3}\right\rceil$ by constructing a resolving set $M\ in\ AWJ_{2n}$ having $\left\lceil\frac{n+1}{3}\right\rceil$ vertices. For this we consider the following cases:

**Case 1:** When $n \equiv 0\ (mod\ 6)$, then we may write $2n = 6k$, where $k \geq 6$ and $\left\lceil\frac{n+1}{3}\right\rceil = k + 1$. In this case, $M = \{v_1, v_{10}\} \cup \{v_{12i+14}, v_{12i+18}: 0 \leq i \leq \frac{k-4}{2}\} \cup \{v_{2n-2}\}$.

**Case 2:** When $n \equiv 1\ (mod\ 6)$, then we may write $2n = 6k + 1$, where $k \geq 6$ and $\left\lceil\frac{n+1}{3}\right\rceil = k + 1$. In this case, $M = \{v_1, v_{10}\} \cup \{v_{12i+14}, v_{12i+18}: 0 \leq i \leq \frac{k-4}{2}\} \cup \{v_{2n-2}\}$.

**Case 3:** When $n \equiv 2\ (mod\ 6)$, then we may write $2n = 6k + 4$, where $k \geq 6$ and $\left\lceil\frac{n+1}{3}\right\rceil = k + 1$. In this case, $M = \{v_1, v_{10}\} \cup \{v_{12i+14}, v_{12i+18}: 0 \leq i \leq \frac{k-4}{2}\} \cup \{v_{2n-2}\}$.

**Case 4:** When $n \equiv 3\ (mod\ 6)$, then we may write $2n = 6k$, where $k \geq 5$ and $\left\lceil \frac{n+1}{3} \right\rceil = k + 1$. In this case, $M = \{v_1, v_{10}\} \cup \{v_{12i+14}, v_{12i+18}: 0 \leq i \leq \frac{k-5}{2}\} \cup \{v_{2n-4}, v_{2n}\}$.

**Case 5:** When $n \equiv 4\ (mod\ 6)$, then we may write $2n = 6k + 2$, where $k \geq 5$ and $\left\lceil \frac{n+1}{3} \right\rceil = k + 1$. In this case, $M = \{v_1, v_{10}\} \cup \{v_{12i+14}, v_{12i+18}: 0 \leq i \leq \frac{k-3}{2}\}$.

**Case 6:** When $n \equiv 5\ (mod\ 6)$, then we may write $2n = 6k + 4$, where $k \geq 5$ and $\left\lceil \frac{n+1}{3} \right\rceil = k + 1$. We define $M = \{v_1, v_{10}\} \cup \{v_{12i+14}, v_{12i+18}: 0 \leq i \leq \frac{k-3}{2}\}$.

The set M contains only one major vertex, rest of the vertices are all minor vertices. So there is a unique 5-4 major and 5-4 minor gap, and rest of all are minor 4-4 gaps containing seven and three, one and three or three and three vertices alternative. M is a resolving set of $AWJ_{2n}$, since any two minor or any two major vertices, respectively, lying in different gaps (neighboring or not) are sepatated by at least one vertex in the set of three or four vertices of M determining these two gaps. This property is true for the vertices lying in the same gap. Also we note that $r(v|S) = (1,2,3, \dots, 2)$ and $r(v|S) \neq r(x|S)$, for every $x \in V(AWJ_{2n})$ where $v$ is a central vertex and $x \neq v$.
To prove that $\beta(AWJ_{2n}) \geq \left\lceil \frac{2n+1}{6} \right\rceil$, Let B be a basis of $AWJ_{2n}$ $and$ $|B| = l$. Then B induces $l$ gaps on $C_{2n}$, namely $g_1, \dots, g_l$ such that $g_j$ $and$ $g_{j+1}$ are neighboring gaps for every $1 \leq j \leq l - 1$, and also $g_1 and\ g\_l$ are neighboring gaps. By Lemma 4.2, at most one of the gaps is major, say $g_1$, My Lemma 4.3, and Lemma 4.4, we can write
$|g_1| + |g_2| \leq 11;$
$|g_2| + |g_3| \leq 6;$
$|g_{l-1}| + |g_l| \leq 9;$
$|g_l| + |g_1| \leq 12$

And
$|g_j| + |g_{j+1}| \leq 10,$

For every $j = 3, \dots, l - 2$. By adding these inequalities, we get
$2(2n - l) = 2\sum_{j=1}^{l} |g_j| \leq 10l - 2.$

It follows that $l \geq \left\lceil \frac{2n+1}{6} \right\rceil$. Since $l$ is an integer, for each $2n \equiv 0,2,4(mod\ 6), we\ have\ l \geq \left\lceil \frac{2n+1}{6} \right\rceil$, which completes the result.

## Conclusion
In this paper, we have determined the metric dimension (location number) of m-level wheel graphs, convex polytope graphs and antiweb gear graphs. We proved that these classes of wheel related graphs have unbounded metric dimension. We also gave a negative answer to an open problem proposed in by providing that the infinite class of convex polytopes $\mathbb{Q}_n$ has unbounded metric dimension. We believe that nay infinite class of convex polytopes generated by wheels will have unbounded metric dimension. It is natural to ask for characterization of wheel related graphs with unbounded metric dimension.

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