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Mathematical stochastic calculus on Riemannian manifolds

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Abstract

Mathematical stochastic calculus on Riemannian manifolds is a burgeoning field that merges differential geometry with probability theory to study stochastic processes on curved spaces. This abstract presents an overview of key concepts and results in this interdisciplinary area.

The foundation of stochastic calculus on Riemannian manifolds lies in extending the notions of stochastic differential equations (SDEs) and stochastic integration from Euclidean spaces to manifolds. Key tools include the theory of connections and parallel transport, which provide a framework for defining covariant derivatives and stochastic flows on manifolds.

Central to the study of stochastic processes on Riemannian manifolds is the development of stochastic parallel transport, which generalizes the concept of parallel transport to incorporate stochastic effects. This allows for the analysis of stochastic differential equations driven by Wiener processes or other types of stochastic processes on curved spaces.

Applications of mathematical stochastic calculus on Riemannian manifolds abound in various fields such as mathematical finance, physics, and biology. For instance, in finance, it enables the modeling of asset prices and portfolio optimization in markets where asset prices evolve on curved spaces. In physics, it provides a framework for describing the behavior of particles subject to random fluctuations in curved spacetimes. In biology, it aids in understanding the dynamics of biological systems evolving on complex manifolds.

Overall, mathematical stochastic calculus on Riemannian manifolds offers a rich theoretical framework for studying stochastic processes in curved spaces, with diverse applications across different scientific disciplines. Ongoing research in this area continues to deepen our understanding of the interplay between geometry and randomness, paving the way for new insights and applications in various fields.

Keywords: Stochastic calculus, Riemannian manifolds, differential geometry

Introduction

Stochastic differential equations in diffusion theory in a d-dimensional Euclidean space R^d with continuous pathway are defined by the fundamental d-dimensional Wiener process W^a (t). On a Riemannian manifold M^d, the fundamental Wiener process is difficult to handle. By using an inadequately posed formulation of a stochastic differential equation, it is not assured that its solution remains on the manifold M^d, which leads to inconsistent results. The key idea in the mathematical concept of diffusion on general d-dimensional Riemannian manifolds M^d (with definite metric signature) is to define a stochastic process on the curved manifold using the fundamental Wiener process, each component of which is a process in the Euclidean space R^d [1-2]. Intuitively, we can understand this concept as follows. Consider a two-dimensional stochastic motion of a particle on a plane. If the trajectory of the particle is traced in ink and a sphere on the plane is rolled along the stochastic curve without slipping the resulting transferred path defines a random curve or a stochastic Markovian process on the sphere. This method can be applied for diffusion on a general Riemannian manifold. The tangent space of a Riemannian manifold is endowed with Euclidean structure and, therefore, we can move the manifold in the tangent space by construction of a parallel translation along the stochastic curve with the help of the orthonormal frame vectors $e_a=e^i_a (x)\ \partial_i$ (i, a=1, . . . , d) and the Christoffel connection coefficients Γ^j_{ib}, $x=(x_1, . . . , x_d)$, $\partial_i=\partial/\partial x^i$. In local coordinates on a Riemannian manifold, the infinitesimal motion of a smooth curve c^i (t) in M^d is that of $\gamma^i(t)$ in the tangent space (which can be identified with R^d) by using a parallel transformation: $dc^i=e^i_a (x)d\gamma^a$ and $de^i_a(x)=-\Gamma^i_{ml}e^l_a dc^m$. Therefore, a random curve can be defined in the same way by using the canonical realization of a d-dimensional Wiener process (defined in the Euclidean space) and substituting $d\gamma^a \rightarrow dW^a(t)$. Thus, the stochastic differential equations describing diffusion on a Riemannian manifold in the orthonormal frame bundle O(M) with coordinates $O(M)=\{x^i, e^i_a\}$ are given by.

$$dx^i(\tau) = e^i_a(\tau)\circ dW^a_\tau + b^i(\tau)d\tau,$$

$$de^i_a(\tau) = -\Gamma^i_{ml}e^l_a \circ dx^m(\tau). \quad (1)$$

Here $\delta^{ab}e^i_a[x(\tau)]e^i_b[x(\tau)]=g^{ij}$, $\partial_i g^j_a=\Gamma^j_{ik}e^k_a$, g^{ij} is the Riemannian metric, and δ^{ab} is the flat Euclidean metric, where δ^{ab} is the Kronecker symbol. The components of the elementary Wiener process $dW^a=W^a(t+\Delta t)-W^a(t)$ are defined in the Euclidean space with the probability density $P(W^a) = \left(-\frac{[w^a(t)]^2}{2D\Delta t}\right)$ and with the expectation values $\langle W^a\rangle=0$ and $\langle W_a(\tau)W^b(\tau+s)\rangle=Ds\delta_{ab}$. b^i's are the components of an arbitrary tangential vector and D is the diffusion constant, which here is independent on the time and space variables. Equation (1) is defined in the Stratonovich calculus (denoted by the symbol ∘).

Associated to each diffusion process, there is a secondorder differential operator denoted as the generator A of the diffusion. Diffusion processes in a d-dimensional Euclidean space are described by stochastic differential equations of the form,

$$dX^i = \sigma^i_a(\tau,X)\, dW^a + b^i(\tau,X)d\tau\ . \quad (2)$$

$X=(X_1, \ldots, X_d)\in R^d$ is a stochastic process with $X(0)=x$; $x=(x_1, \ldots, x_d)$, and τ is the time ($\tau\geq 0$). The diffusion coefficients $\sigma^i_a(\tau,X)$ are given matrices and the drift coefficients $b^i(\tau,X)$ are coefficients of a smooth vector field. W^a's are the components of the elementary Wiener process. Equation (2) can be transformed into an integral equation.

$$X^i_\tau = X^i_0 + \int_0^\tau \sigma^i_a(s,X)dW^a_s + \int_0^\tau b^i(s,X)ds. \quad (3)$$

The stochastic integral in the second term of Eq. (3) is defined as the limit $\int_0^\tau \sigma^i_a(s,X)dW^a_s = \sum_{i=n}^{n} \sigma^i_a(s^*_i,X)[W^a(s_i)-W^a(s_{i-1})]$ as $n\rightarrow\infty$. This integral depends on the choice of the intermediate point s^*_i. With the choice $s^*_i=s_{i-1}$ (postpoint rule), the Ito stochastic integral is defined. The Ito integral is a Markovian process and plays a fundamental role in the theory of diffusion processes and most of mathematical treatments can only rigorously proven by using this calculus. Alternatively, choosing $s^*_i=s_{i-1}$ (midpoint rule) the Stratonovich stochastic integral is defined. The Stratonovich integral has the advantage of leading to ordinary chain rule formulas under a transformation. This property makes the Stratonovich integral natural to use for stochastic differential equations on Riemannian manifolds. However, in general Stratonovich integrals are not Markovian processes, which hinders rigorous mathematical treatments in most cases. Note that the chosen interpretation has to be denoted in the differential equation. The symbol $\sigma^i_a(\tau, X)dW^a$ implies the Ito integral interpretation and $\sigma^i_a(\tau,X)\circ dW^a_\tau$ the Stratonovich interpretation.

With the Ito interpretation, the solution X^i_τ of Eq. (2) is denoted as an Ito process if the diffusion and drift coefficients satisfy the Lipschitz condition, and $\sigma^i_a(\tau,X)$ is adapted to the fundamental Wiener process W^a_τ. An Ito process has the important property of being Markovian. Then $Y_\tau = f(X_\tau)$ is also an Ito process. Associated to an Ito process is the diffusion generator A of X_τ, which is defined to act on a suitable function f by.

$$Af = \lim_{t\rightarrow 0}\frac{E^W[f(X_\tau)]-f(x)}{t}, \quad (4)$$

Where $x=X_o$ is the initial point of X_τ. For the stochastic process described by Eq. (2), A is given by [3-4].

$$Af = \frac{D}{2}\delta^{ab}\sigma^i_a(\tau,X)\, \sigma^i_b(\tau,X)\partial_i\partial_i f + b^i(\tau+X)\partial_i f \quad (5)$$

The generator A describes how the expected value $u(t,x)=E^x[f(X_\tau)]$ of any smooth function f of X evolves in time and satisfies the following equation.

$$\frac{\partial}{\partial\tau}u(\tau,x) = Au(\tau,x), \quad (6)$$

With $u(0,u)= f(x)$. Equation (6) is denoted as the Kolmogorov's backward equation. The Fokker-Planck equation (or forward Kolmogorov equationdescribes how the probability density function $\phi(\tau,x)$ of X_τ evolves with time. The probability density function can be used to calculate the expected value $E^x[f(X_\tau)]$ by $E^x[f(X_\tau)] = f_\Omega(x)\phi(\tau,x)dx_1\ldots dx_d$, where Ω in the domain of the d-dimensional space of the variables $X_1, . . , X_d$. The Fokker-Planck equation within the Ito integral interpretation is given by the following equation.

$$\frac{\partial}{\partial \tau}\phi(\tau,x) = A^*\phi(\tau,x), \tag{7}$$

With the adjoint operator A^*.

$$A^* f = \frac{D}{2}\delta^{ab}\partial_i\partial_j\sigma_a^j(\tau,X)f - \partial_i b^i(\tau,X)f. \tag{8}$$

Since the stochastic calculus on Riemannian manifolds is naturally formulated in the Stratonovich integral interpretation, we will consider the connection between both types of integrals. Let us formulate the stochastic differential equation (2) with the Ito interpretation by a corresponding equation with the Stratonovich interpretation.

$$dX^i = \tilde{\sigma}_a^i(\tau,X)\circ dW^a + \tilde{b}^i(\tau,X)d\tau. \tag{9}$$

There exists a connection between Ito and the Stratonovich integrals [5-6], which allows to associate the diffusion and drift terms in one of the interpretations with the other.

$$\begin{gathered}\tilde{b}^i(\tau,X) = b^i(\tau,X) - \delta^{ab}\frac{D}{2}\sigma_a^j(\tau,X)\partial_i\sigma_b^i(\tau,X)\\ \sigma_a^i(\tau,X) = \tilde{\sigma}_a^i(\tau,X).\end{gathered} \tag{10}$$

Substituting $\tilde{b}^i(\tau,X)$ into Eq. (5), the diffusion operator A in the Stratonovich interpretation is.

$$A = \frac{D}{2}\delta^{ab}\sigma_a^i(\tau,x)\partial_i\sigma_b^i(\tau,x)\partial_j + \tilde{b}^i(\tau,x)\partial_i. \tag{11}$$

The Fokker-Planck equation (7) with respect to the Stratonovich interpretation is then given by.

$$\begin{aligned}\frac{\partial}{\partial \tau}\phi(\tau,x) = {} & \frac{D}{2}\delta^{ab}\partial_i\sigma_a^i(\tau,x)\partial_i[\sigma_b^i(\tau,x)\phi(\tau,x)]\\ & -\partial_i\tilde{b}^i(\tau,x)\phi(\tau,x).\end{aligned} \tag{12}$$

Introducing the fundamental vector fields.

$$L_a = \sigma_a^j(\tau,x)\partial_i, L_0 = \tilde{b}^i(\tau,x)\partial_i, \tag{13}$$

The generator A of the stochastic process in the Stratonovich interpretation can be expressed by the operators L_a, L_0 as follows.

$$A = \frac{D}{2}\delta^{ab}L_aL_b + L_0. \tag{14}$$

Equation (14) is an important formula for the calculus on Riemannian manifolds. On a Riemannian manifold, the driving Wiener process W^a of a stochastic differential equation is difficult to handle. In differential geometry for a general d-dimensional Riemannian manifold M^d (with definite metric signature) equipped with a Christoffel connection Γ^j_{ib}, it is possible to lift a smooth curve $c^i(t)$ in M^d to a horizontal curve in the tangent bundle TM (which is endowed with a Euclidean structure) by using the bundles of orthonormal frames $e_a=e^i_a(x)\partial_i(i,a=1, . . , d)$. The orthonormal frame bundle O(M) is described by the local coordinates $\{r=(x^i,e^i_j)\}=O(M)$. The infinitesimal motion of a smooth curve $x^i(t)$ in M^d is that of $\gamma^i(t)$ in O(M) described by the ordinary differential equations for a parallel transport.

$$dx^i = e^i_a(x)d\gamma^a,$$

$$de^i_a(x) = -\Gamma^i_{ml}e^l_a dx^m. \tag{15}$$

Here $\delta^{ab}e^i_a(x)e^i_a(x)=g^{ij}$, $\partial_i e^j_a = -\Gamma^j_{ib}e^b_a$ g^{ij} is the Riemannian metric and δ^{ab} is the flat Euclidean metric, where δ^{ab} is the Kronecker symbol. $r^i(t)$ is called the horizontal lift of the curve $x^i(t)$ to the orthonormal frame bundle O(M) and it lies in the Euclidean space R^{d+d2}. The horizontal curve $\gamma^i(t)$ corresponds uniquely to a smooth curve in the tangent space (which can be identified with an Euclidean space R^d). Correspondingly, a random curve can be defined in the same way as in Eq. (15) by using the canonical realization of a d-dimensional Wiener process and substituting $d\gamma^a \rightarrow dW^a(t)$. Therefore, the stochastic differential equation describing diffusion on a Riemannian manifold is [7-8].

$$dx^i = e^i_a(\tau) \circ dW^a + b^i d\tau,$$

$$de^i_a(\tau) = -\Gamma_{ml}e^1_a(\tau) \circ dx^m, \tag{16}$$

Where the components of an arbitrary tangential vector b^i are additionally introduced for a more general situation with an account of an external force field.

The derivation of the Kolmogorov backward equation with the definition of the diffusion operator $A_{O(M)}$ can be performed by the same rules, as in Euclidean space in the Stratonovich calculus. Corresponding the definition of the fundamental vector fields L_a and L_0 in Eq. (13), one can now introduce the fundamental horizontal vector fields H_a and H_0 on O(M) for the extended stochastic differential equation system Eq. (16).

$$H_a = e^i_a \frac{\partial}{\partial x^i} - \Gamma_{ml} e^l_a \frac{\partial}{\partial e^i_b}$$

$$H_0 = b^i(\tau,X)\partial_i - \Gamma^i_{ml} e^l_a(\tau) b_m \frac{\partial}{\partial e^i_a}, \tag{17}$$

And the operator $A_{O(M)}$ for the stochastic process in the orthonormal frame bundle is given by.

$$A_{O(M)} = \frac{D}{2}\delta^{ab} H_a H_b + H_0. \tag{18}$$

$A_{O(M)}$ is the horizontal lift of the diffusion generator A_M on the manifold to the orthonormal frame bundle. Obviously, the projection of a function in $O_{(M)}$ to M with f(r) = f(x,0), $r=(x^i,e^i_j)$, satisfies the relation.

$$A_{O(M)}f(r) = A_M f(x), \tag{19}$$

Where $A_M = \frac{D}{2}\delta^{ab}(e^i_a\partial_i e^j_b\partial_j) + b^i\partial_i = \left(\frac{D}{2}\Delta_M + b^i\partial_i\right)$ and $\Delta_M = g^{ij}\partial_i\partial_j + g^{ij}\Gamma^k_{ij}\partial_k$ is the Laplace-Beltrami operator. The generalized Kolmogorov backward equation on a Riemannian manifold is obtained by.

$$\frac{\partial}{\partial\tau}u(\tau,x) = \left(\frac{D}{2}\Delta_M + b^i\partial_i\right)u(\tau,x). \tag{20}$$

The Fokker-Planck operator on a Riemannian manifold cannot be derived directly like in the Euclidean case. But as explained above, this operator is given by the adjoint of the diffusion generator A^* (which includes the volume element $\sqrt{g}$, $g=\det\{g_{ij}\}$). Since the Laplace-Beltrami operator is selfadjoint $\Delta_M=\Delta^*_M$, the generalized Fokker-Planck equation on a Riemannian manifold takes the form.

$$\frac{\partial\Phi}{\partial\tau} = -div_x(b\Phi) + \frac{D}{2}\Delta_M\Phi, \tag{21}$$

Where $div_x(b\Phi)=g^{-1/2}\partial_i(g^{1/2}b^i\Phi)$ is the divergence operator in the Riemannian manifold, $\Phi=\Phi(x,\tau|y,0)$ is the transition probability with the initial condition $\Phi(x,0|y,0)=\delta(x-y)$ and adequate boundary conditions at infinity. The probability density $\varphi(x,\tau)$ is determined by the same equation with the initial condition $\varphi(x,\tau=0)=\varphi^0(x)$.

Corresponding to Eqs. (17) and (18), the diffusion generator $A_{O(M)}$ on O(M) can be projected on M^d with f(r)= f(x,0), $r=(x^i,e^i_j)$ using the relation $A_{O(M)}f(r) = A_M f(x)$, where.

$$A_M = \frac{D}{2}\delta^{ab}(e_a^i\partial_i e_b^i\partial_j) + b^i\partial_i = \left(\frac{D}{2}\Delta_M + b^i\partial_i\right) \tag{22}$$

and $\Delta_M=g^{ij}\partial_i\partial_j-g^{ij}\Gamma^k_{ij}\,\partial_k$ is the Laplace-Beltrami operator on the manifold M^d. The generalized Fokker-Planck equation is obtained by the adjoint of the diffusion generator A^*_M (which includes the volume element.

$\sqrt{g}$, g=det{g^{ij}}). Since the Laplace-Beltrami operator is self-adjoint, $\Delta_M=\Delta^*_M$, this equation takes the form.

$$\frac{\partial\Phi}{\partial\tau} = -div_x(b\Phi) + \frac{D}{2}\Delta_M\Phi, \tag{23}$$

Where $div_x(b\Phi=g^{-1/2}\partial_i(g^{1/2}b^i\Phi)$ is the divergence operator on the Riemannian manifold, $\Phi=\Phi(x,\tau|y,0)$ is the transition probability with the initial condition $\Phi(x,0|y,0)=\delta(x-y)$ and adequate boundary conditions at infinity. The probability density $\varphi(x^i,\tau)$ is determined by the same equation with the initial condition $\varphi(x,\tau=0)=\varphi^0(x)$.

A remarkable feature of Markovian diffusion on a Riemannian manifold is the supposition that for the diffusion coefficients in Eq. (1) only the orthonormal frame coefficients $e^i_a(x)$ are admissible, which are directly related to the geometry of the Riemannian manifold. In contrast, on Euclidean manifolds, a much more general class of diffusion coefficients are permitted.

Conclusion

The field of mathematical stochastic calculus on Riemannian manifolds integrates differential geometry with probability theory to analyze stochastic processes on curved spaces. By extending stochastic differential equations (SDEs) and stochastic integration from Euclidean spaces to manifolds, it provides a robust framework for examining random processes in diverse scientific fields.

Key elements of this calculus include the theory of connections and parallel transport, which facilitate the definition of covariant derivatives and stochastic flows on manifolds. The stochastic parallel transport is particularly central, allowing for the generalization of parallel transport to include stochastic effects.

Applications of this advanced mathematical framework are widespread, ranging from modeling asset prices and portfolio optimization in finance to describing particle behavior in curved spacetimes in physics and understanding biological system dynamics on complex manifolds in biology.

The methodology employs the Stratonovich and Ito integrals, with a preference for the former due to its natural alignment with differential geometry, despite its non-Markovian nature which complicates rigorous treatments. The research also delves into the connections between these integral interpretations, the formulation of the Kolmogorov backward equation, and the Fokker-Planck equation on Riemannian manifolds, enhancing the theoretical underpinnings and practical applications of stochastic processes in curved spaces.

This intersection of geometry and randomness continues to evolve, offering new insights and potential applications across various scientific disciplines, ultimately deepening our understanding of stochastic processes on Riemannian manifolds.

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